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1993 J. Phys. A: Math. Gen. 26 L719

(http://iopscience.iop.org/0305-4470/26/16/002)

J. Phys. A: Math. Gen. 26 (1993) L719–L722. Printed in the UK

LETTER TO THE EDITOR

Exact solution to a toy random field model

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Received 16 April 1993

Abstract. A toy random field model in one dimension is solved exactly by a mapping onto a stochastic differential equation.

Many concepts in the physics of disordered systems like metastability, the breakdown of perturbation theory, or the idea of replica symmetry breaking have been tested on one dimensional toy models of random field problems [1–4]. Thus it is useful to find such a model where physical quantities can be calculated exactly and in closed form.

It is the aim of this short letter to give an exact solution for the probability distribution of the partition function for a specific one-dimensional model by relating it to a stochastic differential equation.

We consider the partition function

$$Z=\int_{-\infty}^{\infty} \mathrm{d}x \exp\{-\beta(V(x)+\sigma W(x))\} \tag{1}$$

β being the inverse temperature.

We assume that the total Hamiltonian is the sum of a deterministic potential $V(x)$ and a random function

$$W(x)=\int^{x} \xi(x')\,\mathrm{d}x' \tag{2}$$

where $\xi(x)$ are uncorrelated Gaussian random variables, i.e. $\overline{\xi(x)\xi(x')}=\delta(x-x')$. Thus the random potential is realized by a one-dimensional random walk, and $W(x)$ is the Wiener process.

Equation (1) is a one-dimensional toy model for an interface in a random field Ising ferromagnet, because up to a (possibly infinite) random additive constant we have

$$W(x)\simeq\tfrac{1}{2}\int_{-\infty}^{\infty} \sin(x-x')\xi(x')\,\mathrm{d}x'. \tag{3}$$

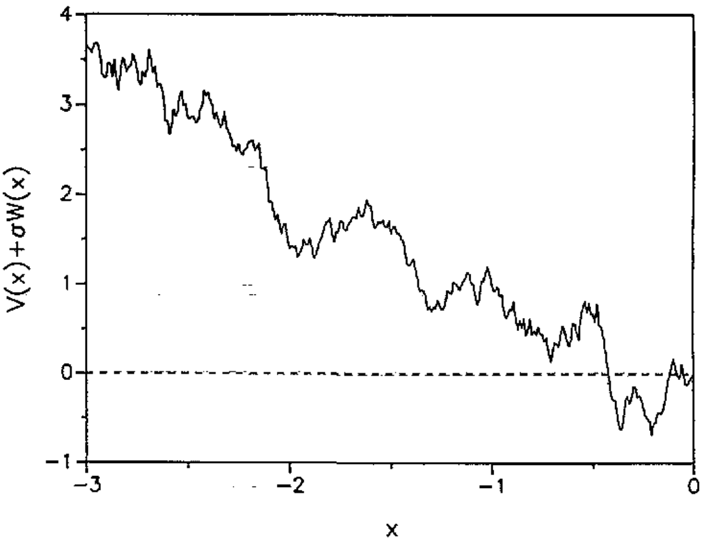

Figure 1. Plot of a realization of the total potential with parameters $\beta=\mu=\sigma=1$. The random potential is zero at $x=0$. The dashed line is the line of zero energy.

This is the energy of a domain wall located at position x under the influence of random fields $\xi(x')$.

In this letter I choose $V(x)$ to be the piecewise linear potential

$$V(x)=\begin{cases}-\mu x & \text{for } x<0\\ \infty & \text{for } x>0.\end{cases} \tag{4}$$

Hence the total Hamiltonian $V(x)+\sigma W(x)$ is a random walk with constant negative drift (figure 1).

In this case the partition function is related to the solution of a well known stochastic differential equation, the so called *Verhulst equation* [5] which reads

$$(\mathrm{d}/\mathrm{d}x)y+y^2-\beta(\mu-\sigma\xi(x))y=0. \tag{5}$$

Here x *plays the role of the time.* Assuming that this multiplicative noise process is interpreted in Stratonovich's sense [6], we can use the transformation $y(x)=1/Z(x)$ to obtain the simpler linear equation

$$(\mathrm{d}/\mathrm{d}x)Z+\beta(\mu-\sigma\xi(x))Z=1 \tag{6}$$

which has the solution

$$Z(x)=\exp(-\beta\mu x)\int_{-\infty}^{x}\mathrm{d}x'\,\exp\{\beta(\mu x'-\sigma W(x')+\sigma W(x))\}. \tag{7}$$

Obviously $Z(x=0)$ equals the desired partition function

$$Z=\int_{-\infty}^{0}\mathrm{d}x'\,\exp\{\beta(\mu x'-\sigma W(x')+\sigma W(0))\} \tag{8}$$

when we assume, that the random potential is zero at $x=0$.

To find the distribution of Z, note that after a transient period at early times $x \to -\infty$, $Z(x)$ is a stationary process. Thus the distribution function of $Z(x)$ does not depend on time and can be calculated from the Fokker–Planck equation for (5) or (6).

This has already been done for the Verhulst equation (5) and the solution can be found in the literature [5]. Nevertheless let me give a simple derivation of the result.

Introducing the free energy $F(x) = -\beta^{-1} \ln Z(x) = \beta^{-1} \ln y(x)$, equation (5) is transformed into

$$\frac{\mathrm{d}}{\mathrm{d}x} F = -\frac{\partial}{\partial F}\{\beta^{-2}\exp(\beta F) - \mu F\} - \sigma\xi(x). \tag{9}$$

This represents the overdamped motion of a particle in the 'Toda' potential $U(F) = \beta^{-2}\exp(\beta F) - \mu F$ being driven by additive noise of amplitude σ. From (9) we can immediately read off the stationary probability density $p(F)$:

$$p(F) = \mathcal{N}^{-1}\exp(-2\sigma^{-2}U(F)) = \mathcal{N}^{-1}\exp(-2\beta^{-2}\sigma^{-2}e^{\beta F} - 2\sigma^{-2}\mu F). \tag{10}$$

The normalization is given by

$$\mathcal{N} = \beta^{-1}\left(\frac{\beta^2\sigma^2}{2}\right)^{2\mu/\beta\sigma^2} \cdot \Gamma\left(\frac{2\mu}{\beta\sigma^2}\right) \tag{11}$$

where $\Gamma(u)$ is Euler's gamma function.

The density $g(Z)$ for the partition function is then simply

$$g(Z) = \mathcal{N}^{-1}\beta Z^{-1(1+2\mu/\beta\sigma^2)} \cdot \exp(-2/\beta^2\sigma^2 Z). \tag{12}$$

This density has a long tail for $Z \to \infty$ (figure 2), which leads to the divergence of the positive moments $\overline{Z^a}$ for all $a > 2\mu/\beta\sigma^2$.

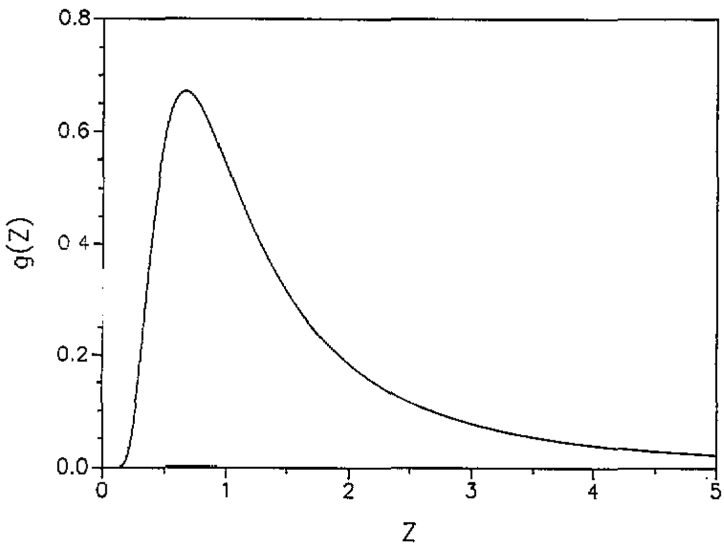

Figure 2. Probability density of the partition function Z for $\beta = \mu = \sigma = 1$.

For the sake of completeness I give the explicit expression for the disorder averaged free energy

$$\overline{F}=\beta^{-1}\ln\left(\frac{\beta^2\sigma^2}{2}\right)+\beta^{-1}\psi\left(\frac{2\mu}{\beta\sigma^2}\right) \tag{13}$$

and the average position of the domain wall at temperature β^{-1}:

$$\overline{\langle x\rangle}=-\frac{\partial}{\partial\mu}\overline{F}=\frac{2}{\beta^2\sigma^2}\psi'\left(\frac{2\mu}{\beta\sigma^2}\right). \tag{14}$$

The brackets denote the thermal average and $\psi(u)=(\partial/\partial u)\ln(\Gamma(u))$.

Simplified expressions are obtained for zero temperature ($\beta=\infty$). Now the free energy equals the absolute minimum E of the potential. From (9) we get the exponential distribution

$$p(E)=\begin{cases}(2\mu/\sigma^2)\exp(2\mu E/\sigma^2) & \text{for } E<0\\ 0 & \text{for } E>0\end{cases}$$

together with $\overline{\langle x\rangle}=-\sigma^2/2\mu$.

Can we apply the present approach to *nonlinear* deterministic potential such as the quadratic $V(x)\propto x^2$, a case which was frequently studied [1–4]? If we again assume that $V(x)=\infty$ for $x>0$ the stochastic differential equation (5) is easily modified (replace μ by $-(\mathrm{d}/\mathrm{d}x)V(x)$ in (5) and (7)). But then the differential equation will contain the time x explicitely. A probability distribution for Z would have to be calculated from a *time dependent Fokker–Planck* equation, which in most cases cannot be solved in closed form.

I would like to thank S Diederich for many inspiring discussions.

References

[1] Engel A 1985 *J. Physique Lett.* **46** L409
[2] Engel A *Preprint*
[3] Villain J, Semeria B, Lancon F and Billard L 1983 *J. Phys. C: Solid State Phys.* **16** 6153
[4] Mezard M and Parisi G *Preprint* LPTENS 92/12
[5] Horsthemke W and Lefever R 1984 *Noise-induced Transitions* (Berlin: Springer)
[6] Gardiner C W 1983 *Handbook of Stochastic Methods* (Berlin: Springer)